1 Graph Coloring

Definition 1. *Given a graph G, a* valid coloring *of G is a set of labels (or colors) for each of the vertices such that no vertex has the same label as any vertex it shares an edge with.*

In general, the *graph coloring problem* is to find a the smallest set of colors such that a valid coloring exists for a given G. We call this minimum number of colors the *chromatic number* of G, or $\chi(G)$.

The naïve approach to solving this would be to use a greedy approach — for each vertex, give color with a new color if and only if there are no free colors. In the worst case, this would use a number of colors equal to the highest degree of the vertices in the graph (where degree is defined as the number of edges that connect to a given vertex).

A *clique* in a graph is a group of vertices such that all of the vertices are connected to each other. While it is clear that the chromatic number of a graph cannot be less that the size of the largest clique, clique size is not the only important factor. In fact, there exist *triangle free* graphs (ie, graphs such that there are no cliques of size 3 or greater) with arbitrarily large chromatic number. Several constructions for building these graphs exist; for further reading, consult the paper *Triangle Free Graphs and Their Chromatic Numbers* by Tengren Zhang.

1.1 The 4-color Problem

The *4-color Problem* says that any planar region that has been divided up into parts can be colored using only 4 colors. (The obvious example of this being maps) This particular theorem was proven by Kenneth Appel and Wolfgang Haken in 1976, and it was the first proof that had a computer-verification aspect to it.

While this document does not have the space to reproduce that proof, it is possible to show a weaker version— specifically, the 6-color Problem.

First, we transform the map into a graph with each vertex representing a region, and edges between each two bordering regions on the map.

If the graph is not triangulated - ie, each of the regions constructed by the graph do not have exactly 3 edges, we can add edges until we have a triangulated graph. (This does not change our solution, as any coloring that solves the triangulated graph will be a solution for the un-triangulated one)

Now, we know that by Euler's formula, $v - e + f = 2$, where v, e, and f are the number of vertices, edges and faces of the graph. Since in our triangular graph, each region is triangular and each edge is shared by two regions, we know that $2e = 3f$, which implies that $3v - 3e + 2e = 3v - e = 6 \Rightarrow 6v - 2e = 12$.

Let v_n be the number of vertices of degree n, and let D be the maximum degree of all vertices of the graph. It follows that

$$6v - 2e = 6\sum_{i=1}^{D} v_i - \sum_{i=1}^{D} iv_i = 12$$

since the degrees of each vertex is equal to twice the number of edges. It follows that

$$\sum_{i=1}^{D}(6-i)v_i = 12$$

and so, since $(6-i) \leq 0$ if $i \geq 6$, there has to be at least one vertex of degree 5 or less.

We can show that any graph can be colored using 6 colors by induction. First, notice that if there's 6 or less vertices, then we can trivially color the graph using 6 colors or less. Next we form our inductive hypothesis: specifically, that any planar graph with n nodes can be colored using six colors. We then take a graph with $n+1$ nodes and show that we can color it as well. First, notice that by our previous result, there exists a node of degree 5 or less. If we remove it from the graph, we get a graph with n nodes, which we can color by our induction hypothesis. Since the node we removed has degree 5 or less, we know that there will be at least one leftover color for us to color that vertex with. Therefore, any planar graph can be colored using 6 colors.

From this, there is a clear algorithm that we can use to color this graph—first, remove any nodes with degree 5 or less from the graph. Then, repeat this process on the resulting graph until we have a graph with 6 or less vertices. From there, we color that graph, then add the nodes back in in the reverse order that we removed them, coloring them as we go.