Math 101 Portfolio

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Overall, I believe my portfolio has a good diversity, I solved four Putnam problems, and several other problems that I found as hard or even harder than the Putnam ones such as OC7, IC42, IC121, IC118, and OC88.
Although I remember working on problems that had to do with counting in two different ways, and the principle of inclusion exclusion, I think none of them made it to my portfolio. However, all other strategies, tactics and tools above have at least one solution.
I am very fond of geometry, which is definitely reflected here. Nevertheless, I had the most fun solving problems related to graph theory, and generating functions. Furthermore, congruence and divisibility have always been challenging to me so I am proud that I got some of those in as well.
Besides all this, there are a few solutions in which I listed "None" for the tactics. Most of these solutions were solved by either drawing a diagram or finding a pattern through observation. Many of them were also proved with induction or contradiction, but I am not sure if that means there is a correlation.

1 IC 7

This solution was submitted only once.
Comments were received on the week of **October 3rd**.
That same week it was given a C^3, along with a challenge, I did not complete the challenge.

Strategies Used

- Get your hands dirty- check each possibility
- Formulate intermediate goals- find the count for each possibility individually

Tactics Used

- Invariance- the sum up to 20 never changes.

Tools Used

- Pascal's triangle and binomial theorem- couting using choose.

Question: How many ways are there to express 20 as a sum of 1's and 2's where the order counts?

This problem is easily illustrated by the following table:

number of 2's in sum	0	1	2	3	4	5	6	7	8	9	10
number of 1's in sum	20	18	16	14	12	10	8	6	4	2	0
total elements to sum	20	19	18	17	16	15	14	13	12	11	10
choose where to position the 2's	$\binom{20}{0}$	$\binom{19}{1}$	$\binom{18}{2}$	$\binom{17}{3}$	$\binom{16}{4}$	$\binom{15}{5}$	$\binom{14}{6}$	$\binom{13}{7}$	$\binom{12}{8}$	$\binom{11}{9}$	$\binom{10}{10}$
number of ways to position the 2's	1	19	153	680	1820	3003	3003	1716	495	55	1

Total number of ways to express 20 as a sum of 1's and 2's: $1 + 19 + 153 + 680 + 1820 + 3003 + 3003 + 1716 + 495 + 55 + 1 =$ **10945**

2 IC 8

This solution was submitted only once.
Comments were received on the week of **October 3rd**.
That same week it was given a C^3.

Strategies Used

- Formulate intermediate goals- get each of the segments of the path separately.

Tactics Used

- Find and exploit symmetries- shortest distances of each of the points to the circle are the same.

Tools Used

- Geometry- Made a diagram of circle to find shortest path.

Question: In the xy-plane, what is the length of the shortest path from $(0,0)$ to $(12,16)$ that does not go inside the circle $(x-6)^2+(y-8)^2=25$?

First notice that the straight line that passes through the origin, $(0,0)$, and the center of the circle, $(6,8)$, also passes through the point $(12,16)$, and the equation of this line is $y=\frac{4}{3}x$. Using this information and the equation of the circle we can know that the points where this line intersects the circle are $(3,4)$ and $(9,12)$. (For more clariry see attatched diagram).

Now, We can create the shortest path desired by drawing two tangents of the circle: one passing through the origin and touching the circle at point a, and the other passing through $(12,16)$ and touching the circle at point b. Then, the path consists of the straight line from the origin to a, followed by the arc along the circle from a to b, and then the straight line from b to $(12,16)$.

To find the dimensions of this path we will use the triangle with vertices at the origin, the center of the circle, and point a. Notice that this triangle is congruent to that with vertices at $(12,16)$, the center of the circle, and point b, so the distance between the origin and a is the same as the distance between b and $(12,16)$. Namely, this distance is $5\sqrt{3}$ (found with trigonometry).

As to the length of the arc from a to b, from the triangle mentioned above we know that the angle of the arc from $(3,4)$ to a equals the angle of the arc from b to $(9,12)$ equals $\frac{\pi}{3}$ and because they both lie on the same line as the angle of the arc from a to b, we know that this angle must also be $\frac{\pi}{3}$, so the length of the arc must be $\frac{5\pi}{3}$.
Therefore the length of the path we are looking at is $5\sqrt{3}+\frac{5\pi}{3}+5\sqrt{3}=10\sqrt{3}+\frac{5\pi}{3}$, and this is the shortest length possible.

3 OC 7

This solution was submitted only once.
Comments were received on the week of **October 3rd**.
That same week it was given a C^3.

Strategies Used

- Generalization- generalized to a sequence of k elements to find a solution for a sequence of 15 elements.
- Get your hands dirty- before finding the pattern I hade to write many 1s and 0s in my notebook.

Tactics Used

- None of those listed in the portfolio template.

Tools Used

- Recurrence relations- Fibonacci.

Question: A sequence $a_1a_2...a_n$ of 0's and 1's is called 1-separated if for no i is $a_ia_{i+1} = 11$, in other words, no two consecutive symbols are 1. Determine the number of 1-separated sequences of length 15.

To solve this problem we notice that each 1-separated sequence of length k can start with either a 1 or a 0. If it starts with a 1, the next element of the sequence (if any) is necessarily 0, and the next $k-2$ elements in the sequence (if any) can be any of the 1-separated sequences of length $k-2$. In the other hand if the length k sequence starts with a 0, the next $k-1$ elements in the sequence (if any) can be any of the 1-separated sequences of length $k-1$.
Knowing this we can come up with a recursive rule. Starting with $k = 1$, there are two 1-separated sequences of length 1, namely 1 and 0. Now for $k = 2$ there are three 1-separated sequences of length 2, namely 00, 01, and 10. Next, for $k = 3$ a 1-separated sequence could start with 0 followed by all 1-separated sequences of length 2 (so there are 3 options here), or it could start with 10 followed by all the 1-separated sequences of length 1 (so there are 2 options here). If we add all the options we get that there are $3 + 2 = 5$ 1-separated sequences of length 3 (000, 100, 001, 101, 010).
Using the same process we will find that the number of 1-separated sequences for $k = 4$ is the ones starting with 0 followed by all options of $k = 3$ plus the ones starting with 10 followed by all the options of $k = 2$, thus there are $5 + 3 = 8$ 1-separated sequences of length 4.
If we continue following this pattern we will soon find that there are $5 + 8 = 13$ 1-separated sequences for $k = 5$, $8 + 13 = 21$ 1-separated sequences for $k = 6$, $13 + 21 = 34$ for $k = 7$ and so on... They are the Fibonacci numbers!! If we look at them closely we can notice that the number of 1-separated sequences of length k is always the $(k+3)^{th}$ Fibonacci number (when the 1st Fibonacci number is taken to be 0).
Therefore the number of 1-separated sequences of length 15 is equal to the 18th Fibonacci number, **1597**.

4 IC16

This solution was submitted only once.
Comments were received on the week of **October 10th**.
That same week it was given a C^3.

Strategies Used

- Relax conditions- make it a 2D problem instead of a 3D problem.

Tactics Used

- None of those found in the portfolio template.

Tools Used

- Geometry- hypothenuse of a triangle.

Question: A bug sits on one corner of a unit cube, and wishes to crawl to the diagonally opposite corner. The bug can't fly, so he has to stay on the surface of the cube. What is the length of its shortest path?

If we unfold the cube into a flat cross (see diagram below), then we will realize the the shortest path the bug can take is across the "straight line" formed be the hypothenuse of a right triangle whose legs are 1 and 2 units long. Thus, the shortest path $= \sqrt{1^2 + 2^2} = \sqrt{5}$.

5 OC15

This solution was submitted once on the week of October 10th.
Comments were received on the week of **October 10th**.
It was submitted for a second time on Novemver 4th.
Comments were received on **December 2nd**. It did not get a C^3 but I have done the changes suggested.

Strategies Used

- Wishful thinking- adding zero creatively.

Tactics Used

- None of those listed in the portfolio template.

Tools Used

- Primes and divisibility- divisibility by 6.

Use induction to prove that $7^n - 1$ is divisible by 6 for every natural number n.

Base case: $n = 1$, $7^1 - 1 = 6$ and $6 \mid 6$.

Induction Hypothesis: Assume $6|(7^k - 1)$ for some k.

Inductive Step: Show $6|(7^{k+1} - 1)$. To do this we re-write $7^{k+1} - 1$ as $7(7^k - 1) + 7 - 1$, then because $6 \mid 7(7^k - 1)$ by the Induction Hypothesis, and $6 \mid 7 - 1$ by base case, we can conclude that, $6|(7^{k+1} - 1)$ which is what we wanted to show.

6 IC18

This solution was submitted once on the week of October 10th.
Comments were received on the week of **October 10th**.
It was submitted for a second time on Novemver 4th.
Comments were received on **December 2nd**. In those comments it received a C^3.

Strategies Used

- Find a penultimate step- if we show that for each partition into to sets exactly one satisfies the conditions we will be done.

Tactics Used

- Find and exploit symmetries- of the partitions of the set into two sets.

Tools Used

- Partitions and bijection- found a bijection between the partitions into two sets and the sets with the characteristics we want.

How many subsets of the set $\{1, 2, \ldots, 30\}$ have the property that the sum of the elements of the subset is greater than 232?

First notice that the sum of all the elements of the set $A = \{1, 2, \ldots, 30\}$ is $\frac{30(30+1)}{2} = 465$, also notice that $\frac{465}{2} = 232.5$ which is almost the same as 232. Thus, we could conjecture that half of the subsets of A have the property that the sum of their elements is strictly greater than 232. To show our conjecture is true, consider all the possible partitions of A into two disjoint sets B and C such that $B \cup C = A$. Furthermore, note that there exist $\frac{1}{2}2^{30} = 2^{29}$ such partitions, which is exactly half of all the possible subsets of A. This means that if we can show that exactly one of the subsets in each of the possible partitions of A has a sum strictly greater than 232, we will be done. Now, we know that the sum of all the elements in $A = B \cup C$ is 465 in all cases, and we have the following two possibilities:

- The sum of all the elements in B is strictly greater than 232. Then the sum of all elements in C should be strictly less than 232, because if it was more, the total sum would be more than 465.

- The sum of all elements in B is strictly less than 232. Then the sum of all elements in B' must be strictly greater than 232 because if not the total sum would be less than 465.

Hence, we have shown that exactly one set in each of the possible partitions of A has a sum that is strictly more than 232. Which is what we wanted to show.

7 IC33

This solution was submitted only once.
Comments were received on the week of **October 17th**.
That same week it was given a C^3.

Strategies Used

- Find a penultimate step- cut the hexagon into six pieces of equal size.

Tactics Used

- Find and exploit symmetries- in the hexagon to find six equilateral triangles.
- Pigeon-hole principle- 7 points and 6 slots.

Tools Used

- Geometry- hexagon, and equlateral tirangles.

Question: Seven points are placed inside a regular hexagon with side length 1. Show that at least two points are at most distance one unit apart.

If we connect opposite vertices in the regular hexagon with three straigt lines (passing through the center) we will end up with six triangular slices. Furthermore, all of these triangles will be equilateral triangles with side length 1.
By the pidgeonhole principle two of the seven points inside of the hexagon will fall within the area of the same triangle. Thus, even if they are in opposite vertices of the triangle, their distance cannot be more than 1.

8 IC37

This solution was submitted only once.
Comments were received on the week of **October 17th**.
That same week it was given a C^3.

Strategies Used

- Generalization- Showing for all even shows for 0.

Tactics Used

- Invariance principle- The number of heads up is always odd.

Tools Used

- None of those listed in the portfolio template.

Question: Seven quarters are initially all heads up. On a single move you can choose any four and turn them over (change heads to tails and tails to heads). Is it possible to obtain all tails up after a sequence of such moves?

The answer to this question is NO. It is impossible to have all tails up after any sequence of moves. In other words, it is impossible to have 0 heads up at any moment. Now, notice that 0 is an even number, thus, if we can show that it is impossible to ever have any even number of heads up at any given stage, we will have shown what we want. This in turn means that it suffices to prove that, after any number of moves, there will always be an odd number of heads up. We can show this by induction as follows.
Base 1: initially there are 7 heads up, 7 is odd.
Base 2: after the initial position of 7 heads up, the only possible move is to turn four heads into tails, this results in 3 heads and 4 tails. 3 is odd.
Induction hypothesis: at any given stage n, there is an odd number of heads face up, that is 7, 5, 3 or 1. Note that this also means that there must be an even number of tails: 0, 2, 4 and 6 respectively.
Inductive step: Given the induction hypothesis, stage $n + 1$ also has an odd number of heads. Consider the cases:

- Stage n has 7 heads: Only option is to turn four heads and zero tails, result: 5 heads. Odd.
- Stage n has 5 heads: We can turn 4 heads 0 tails, results in 1 head. Turn 3 heads 1 tail, results in 3 heads. Turn 2 heads 2 tails, results in 5 heads. All odd.
- Stage n has 3 heads: We can turn 3 heads 1 tail, results in 1 head. Turn 2 heads 2 tails, results in 3 heads. Turn 1 head 3 tails, results in 5 heads. All odd.
- Stage n has 1 head: We can turn 1 head 3 tails, results 3 heads. Turn 0 heads and 4 tails results in 5 heads. Both odd.

Thus, any way we look at it stage $n + 1$ will result in an odd number of heads, and by induction this means that there can never be an even number of heads, so we can never 0 heads up, which is the same as saying we can never get all tails up.

9 IC42

This solution was submitted only once. In addition, I presented this solution in class.
Comments were received on the week of **October 17th**.
That same week it was given a C^3.

Strategies Used

- Wishful thinking- recast problem into a graph and considering smaller graph at the end.

Tactics Used

- Pigeon-hole principle- each vertex must have at least some number of edges of the same color.

Tools Used

- Graph theory- reformulate problem as a graph and look at coloring of edges.

Question: For each pair of people in a group of 17, exactly one of the following is true: "they are strangers", "they are friends", "they are enemies". Prove that there must be a trio all of whom are either mutual strangers, mutual friends, or mutual enemies.

For this problem let us visualize the party as a complete graph, where we have 17 vertices representing the people at the party joined by colored edges determining their relationships. In particular, lets say that an edge between two people is red if they are enemies, blue if they are friends, and green if they have never met.
Now, we are trying to prove that there must be at least one triangle in the graph described above such that its three edges are either all red, all green, or all blue.

Given that this is a complete graph, we know that each vertex has degree 16, furthermore, by the pidgeonhole principle we know that any given vertex has at least 6 edges of the same color. Now, let us consider one of the vertices, a. Without loss of generality let us say that a has at least 6 red edges, and that these edges are connected to vertices b, c, d, e, f, and g. If any two of these vertices, say b and c, are connected by a red edge, we will have a triangle $a - b - c$ that is all red and we would be done.

On the other hand, if there are no red triangles containing a, then the complete sub-graph formed by vertices b, c, d, e, f, and g must have only green and blue edges. So to prove our original statement it suffices to prove that a complete graph with 6 vertices and 2-colored edges will always have a triangle in which all edges are the same color.

To do so we proceed in a similar way as before. Each vertex in our sub-graph has degree 5, and there are 2 colors, thus, by the pidgeonhole principle at least 3 edges of any given vertex are the same color. Consider again without loss of generality, that vertex b has 3 blue edges, connecting it to vertices c, d and e. If any pair of these edges, say c and d is connected by a bue edge, we have a blue triangle $b - c - d$ so we are done. Otherwise c, d and e are all connected by green edges, forming a green triangle $c - d - e$ so we are also done.
Hence, there exists a trio in our 17 people party all of whom are either mutual friends, mutual enemies, or mutual strangers.

10 OC32

This solution was submitted only once.
Comments were received on the week of **October 17th**.
That same week it was given a C^3.

Strategies Used

- Wishful thinking- proof by contradiction always involves some wishful thinking.

Tactics Used

- Invariance principle- the sum of the degrees of the vertices in a graph is always even.

Tools Used

- Graph theory- reformulate problem as graph and look at degrees of vertices.

Question: A large house contains a television set in each room with an odd number of doors. There is only one entrance to the house. Show that it is always possible to enter the house and get to a room with television set.

If we picture the rooms in the house as vertices of a connected graph, and the doors between them as its edges, then we will have a graph with one vertex for each room plus another vertex of degree 1 representing the outside of the house.
Now, to show that it is always possible to enter the house and get to a room with a TV set, is suffices to show that at least one of the vertices in the graph representing a room has an odd degree. To do so, we will use contradiction. Let us assume that no vertex in the graph, besides the outside vertex (which must be degree 1), has an odd degree. That is, assume that all the vertices in the graph representing rooms have an even degree. Then, the sum of the degrees of all such vertices must also be even, and if we add the remaining outside vertex of degree one, we will have that the sum of the degrees of all the vertices in the graph is odd.
However, we know that the sum of the degrees of the vertices in a graph is always 2 times the total number of edges, in other words, the sum of the degrees is always even. But above we said that this same sum was odd, so we have reached a contradiction.
Therefore there must be at least one room in the house with an odd number of doors, and because the graph is connected there is always a way to get to a room with a TV set.

11 IC53

This solution was submitted only once.
Comments were received on the week of **October 24th**.
That same week it was given a C^3.

Strategies Used

- Formulate intermediate goals- prove each of the equations to prove triangles are similar.

Tactics Used

- Find and exploit symmetries- notice that the differences on each side are the same to find equality.

Tools Used

- Complex numbers- reformulating problem in complex numbers makes it simpler.
- Geometry- definition fo similar triangles.

Question: Let A, B, C be three non-collinear points in the plane and M a point in the plane such that it doesn't lie on any of the lines AB, AC, BC. Prove that the centroids of the triangles, MAB, MAC, MBC form a triangle similar to ABC.

Let A, B, C and M be represented by the complex numbers $a_1 + a_2 i$, $b_1 + b_2 i$, $c_1 + c_2 i$ and $m_1 + m_2 i$ respectively. Furthermore, let A', B' and C' be the centroids of the triangles BCM, ACM and ABM respectively. Then, we can say that:

$$A' = \frac{B + C + M}{3} = \frac{b_1 + c_1 + m_1}{3} + \frac{b_2 + c_2 + m_2}{3} i$$

$$B' = \frac{A + C + M}{3} = \frac{a_1 + c_1 + m_1}{3} + \frac{a_2 + c_2 + m_2}{3} i$$

$$C' = \frac{A + B + C}{3} = \frac{a_1 + b_1 + m_1}{3} + \frac{a_2 + b_2 + m_2}{3} i$$

Now, our goal is to show that triangles ABC and $A'B'C'$ are similar triangles. To do so, we must show that

$$\frac{|AB|}{|AC|} = \frac{|A'B'|}{|A'C'|}$$

$$\frac{|AB|}{|BC|} = \frac{|A'B'|}{|B'C'|}$$

$$\frac{|BC|}{|AC|} = \frac{|B'C'|}{|A'C'|}$$

We will start by proving the first of these equations. Using the information above, we can express

$$|AB|/|AC| = \frac{\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}}{\sqrt{(a_1 - c_1)^2 + (a_2 - c_2)^2}}$$

and

$$|A'B'|/|A'C'| = \frac{\sqrt{(\frac{b_1 - a_1}{3})^2 + (\frac{b_2 - a_2}{3})^2}}{\sqrt{(\frac{c_1 - a_1}{3})^2 + (\frac{c_2 - a_2}{3})^2}}$$

Now, if we square both sides of the equation and multiply both sides by $\frac{9}{9}$ we will get the simplified equation

$$\frac{(a_1 - b_1)^2 + (a_2 - b_2)^2}{(a_1 - c_1)^2 + (a_2 - c_2)^2} = \frac{(b_1 - a_1)^2 + (b_2 - a_2)^2}{(c_1 - a_1)^2 + (c_2 - a_2)^2}$$

By observation we can notice that each of the differences on the left hand side of the equation correspond in magnitude with those in the right hand side of the equation, so by squaring them we will obtain the same positive numbers, and thus the ratios are equivalent. It suffices to say that equations

$$\frac{|AB|}{|BC|} = \frac{|A'B'|}{|B'C'|}$$
$$\frac{|BC|}{|AC|} = \frac{|B'C'|}{|A'C'|}$$

are proved in a similar way. Therefore the triangles ABC and $A'B'C'$ are similar triangles.

12 IC54

This solution was submitted only once.
Comments were received on the week of **October 24th**.
That same week it was given a C^3.

Strategies Used

- Formulate intermediate goals- find the sum of coefficients, find the desired pairs of terms, etc.

Tactics Used

- Find and exploit symmetries- in the binomial expansion.

Tools Used

- Generating functions- to generate desired coefficients.
- Pascals triangle and the binomial theorem- that's all the question is.

Question: Prove that for any positive integer n:

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

To solve this let us consider the generating function $(x+1)^n(x+1)^n = (x+1)^{2n}$. By the binomial theorem we know that the expansion of $(x+1)^n$ is

$$\binom{n}{0}1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$$

and we know that this expansion squared must equal $(x+1)^{2n}$. Furthermore, we know that the coefficient of x^n in the expansion of $(x+1)^{2n}$ must be $\binom{2n}{n}$ which is exactly the number we are interested in. Thus, if we add all the coefficients of x^n in the multiplication $(x+1)^n(x+1)^n$ (when both binomials are expanded), we know that it must equal $\binom{2n}{n}$.
This all means that we need to show that the sum of all the coefficients of x^n in the multiplication $(x+1)^n(x+1)^n$ (when both binomials are expanded) is the same as

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2$$

Now, to get x^n we need to have a multiplication of terms that multiplies x^a times x^b such that $0 \leq a, b \leq n$ and $a + b = n$. Therefore, if we have the term x^0 with coefficient $\binom{n}{0}$ it must be multiplied by the term x^n with coefficient $\binom{n}{n}$, if we have the term x^1 with coefficient $\binom{n}{1}$ it must be multiplied by the term x^{n-1} with coefficient $\binom{n}{n-1}$, and so on. In general, if we have the term x^k with coefficient $\binom{n}{k}$ it must be multiplied by the term x^{n-k} with coefficient $\binom{n}{n-k}$ (so the new coefficient would be $\binom{n}{k}\binom{n}{n-k}$).

Next, let us recall that a property of the binomial coefficients is that $\binom{n}{k} = \binom{n}{n-k}$ for any $n \in \mathbb{Z}$ and any $0 \leq k \leq n$. So it is possible to re-write

$$\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}\binom{n}{k} = \binom{n}{k}^2$$

From this it follows that the sum of all the coefficients of x^n in the multiplication $(x+1)^n(x+1)^n$ (when both binomials are expanded) is the same as

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n-1}\binom{n}{1} + \binom{n}{n}\binom{n}{0} =$$

$$\binom{n}{0}\binom{n}{0} + \binom{n}{1}\binom{n}{1} + \cdots + \binom{n}{n-1}\binom{n}{n-1} + \binom{n}{n}\binom{n}{n} =$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2$$

which is what we wanted to show.

13 OC41

This solution was submitted only once.
Comments were received on the week of **October 24th**.
That same week it was given a C^3.

Strategies Used

- Get your hands dirty- performing the multiplications on a table.

Tactics Used

- Extremal principle- look at the smallest number we need and discard other ones to make computation faster.

Tools Used

- Generating functions- the whole solution depends on finding the correct one.

Question: A participant in a contest is rated on a scale of 1 to 7 by each of 4 judges. To be a finalist a participant must score at least 24. Find the number of ways the judges can rate a participant so as to become a finalist.

For this problem we know each of the 4 judges has 7 options of scores he or she can give. We could imagine that to give a score, each of the judges has to pick one term out of the polynomial $x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x$ with an exponent matching the score they want to give, then the multiplication of those terms would be the total score of the participant. Then we can use the generating function $f(x) = (x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x)^4$ to know how many possible ways there are for a participant to get a certain score by looking at the coefficient of the exponent we are interested in. For example, if we want to know how many possible ways a participant could get 17 points, we would look at the coefficient of x^{17} in $f(x)$.
For a participant to be a finalist they have to score 24 or more points. So to know in how many ways a finalist score can happen, we need to add the coefficients of the terms $x^{28}, x^{27}, x^{26}, x^{25}$ and x^{24} in $f(x)$.
To find these coefficients consider the function $g(x)$ such that $g(x)^2 = f(x)$, then we know that

$$g(x) = (x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x)^2$$

and after some computation (see tables attatched) we get that

$$g(x) = x^{14} + 2x^{13} + 3x^{12} + 4x^{11} + 5x^{10} + 6x^9 + 7x^8 + 6x^7 + 5x^6 + 4x^5 + 3x^4 + 2x^3 + x^2 + x$$

Knowing this, we can now compute the first few terms of $g(x)^2 = f(x)$ (see tables attatched). More specifically, the terms of $f(x)$ we care about are $x^{28} + 4x^{27} + 10x^{26} + 20x^{25} + 35x^{24} + \dots$ and the number we are looking for is $1 + 4 + 10 + 20 + 35 = 70$ ways in which a participant could get 24 or more points in the contest.

14 IC83

This solution was submitted only once.
Comments were received on the week of **October 31st**.
That same week it was given a C^3.

Strategies Used

- Wishful thinking: imagine that the claim is true and try to find evidence either way.

Tactics Used

- Pigeon-hole principle- two out of three things must be either 1 or -1.

Tools Used

- Factor tactic- look at the properties of the factors of a function.

- Polynomials- properties of linear equations.

Question: Let a, b, c be distinct integers. Can the polynomial $f(x) = (x-a)(x-b)(x-c)-1$ be factored $f(x)$ into the product of two polynomials of positive degree and integer coefficients?

In short, the answer is no. We can prove this by contradiction. Let us assume that in fact there is a pair of functions $g(x)$ and $h(x)$ such that $f(x) = g(x)h(x)$ and $(x-a)(x-b)(x-c) = g(x)h(x)+1$. From this it follows that

$$g(a)h(a) = -1$$

$$g(b)h(b) = -1$$

$$g(c)h(c) = -1$$

(because a, b and c are zeros of $(x-a)(x-b)(x-c)$). By our assumption we know that the coefficients of both $g(x)$ and $h(x)$ are integers, thus, the only way to get -1 as a result in any of the above multiplications, is by multiplying 1 times -1. That is, if $g(a) = -1$, then $h(a) = 1$ and viceversa, and the same applies for b and c. Furthermore, by the pidgeonhole principle we know that out of $g(a)$, $g(b)$, and $g(c)$ at least two must have the same value. The same applies for $h(a)$, $h(b)$ and $h(c)$.
Now, because the exponents must be positive, we know that to get a polynomial of degree 3 such as $f(x)$, it must be that one of the functions $g(x)$ and $h(x)$ must be quadratic, and the other must be linear. Without loss of generality lets say that $h(x)$ is the linear one. We know that a linear function must have a different y value for each distict vallue of x, we also know that a, b, and c are distinct by definition. But we just stated that at least two out of $h(a)$, $h(b)$ and $h(c)$ have the same value (either 1 or -1). Hence we have reached a contradition, from which it follows that there is no pair of functions $g(x)$ and $h(x)$ such that their product equals $f(x)$.

15 IC89

This solution was submitted only once.
Comments were received on the week of **October 31st**.
That same week it was given a C^3.

Strategies Used

- Find a penultimate step- try to find a way to use $x^2 + y^2 \geq 2xy$.

Tactics Used

- Find and exploit symmetries- to get to the penultimate step.

Tools Used

- Inequalities- it is all about inequalities.

Question: Prove for all real numbers x, y, z that $x^2 + y^2 + z^2 \geq xy + yz + zx$

Notice that because the square of any real number is always positive, we can say that

$$(x - y)^2 \geq 0 \implies x^2 - 2xy + y^2 \geq 0 \implies x^2 + y^2 \geq 2xy$$

Similarly

$$(x - z)^2 \geq 0 \implies x^2 + z^2 \geq 2xz$$

and

$$(y - z)^2 \geq 0 \implies y^2 + z^2 \geq 2yz$$

Furthermore, if we add all the left sides and all the right sides of these inequalities we will get that

$$x^2 + y^2 + x^2 + z^2 + y^2 + z^2 \geq 2xy + 2xz + 2yz$$

$$2x^2 + 2y^2 + 2z^2 \geq 2xy + 2xz + 2yz$$

Thus,

$$x^2 + y^2 + z^2 \geq xy + xz + yz$$

Which is what we wanted to show.

16 IC76

This solution was submitted only once.
Comments were received on the week of **October 31st**.
That same week it was given a C^3.

Strategies Used

- Get your hands dirty- compute the first few terms of the sequence to get a conjecture.
- Formulate intermediate goals- prove for all even and for all odd separately.

Tactics Used

- None of those listed in the portfolio template.

Tools Used

- Arithmetic and geometric sequences and series- at the end it turns out we are dealing with a geometric sequence.
- Recurrence relations- the initial formula is a recurrence relation.

Question: The sequence $a_0, a_1, a_2, \dots$ satisfies the equation

$$a_{m+n} + a_{m-n} = \frac{1}{2}(a_{2m} + a_{2n})$$

for all non-negarive integers m, n with $m \geq n$. If $a_1 = 1$ determine a_n.

Starting with our initial knowledge of the sequence, we can set $m = 1$ and $n = 1$ then

$$a_2 + a_0 = \frac{1}{2}(a_2 + a_2) = \frac{2a_2}{2} = a_2 \implies a_0 = 0$$

Then we can use this information and set $m = 1$ and $n = 0$ to find that

$$a_1 + a_1 = \frac{1}{2}(a_2 + a_0)$$

$$2(1) = \frac{a_2 + 0}{2}$$

$$4 = a_2$$

Following a similar process we can then use $m = 2$ and $n = 0$ to find $a_4 = 16$. $m = 2$, $n = 1$ to find $a_3 = 9$, and so on. With this we soon come to the conjecture that $a_n = n^2$ for any non-negative integer n. We can prove this by induction:
Base cases: $a_0 = 0$, $0^2 = 0$. $a_1 = 1$, $1^2 = 1$. $a_2 = 4$, $2^2 = 4$. $a_3 = 9$, $3^2 = 9$.
Inductive hypothesis: assume $a_n = n^2$ for all non-negative n up to some $n = k$.
Inductive step: Show that the induction hypothesis implies $a_{k+2} = (k+2)^2$. Then our base case of 0 will imply all the even values of n and our base case of 1 will imply all the odd values of n. We can prove this by cases:

- Case 1: k is even:
  In this case we can set $m = \frac{k}{2} + 1$ and $n = \frac{k}{2} - 1$. Then $m + n = k$, $m - n = 2$, $2m = k + 2$ and $2n = k - 2$. We know that all these values (except for $k + 2$) are less than or equal to k. Thus

$$a_k + a_2 = \frac{1}{2}(a_{k+2} + a_{k-2})$$

$$k^2 + 4 = \frac{a_{k+2} + (k-2)^2}{2}$$

$$2k^2 + 8 - (k-2)^2 = a_{k+2}$$

$$2k^2 + 8 - k^2 + 4k - 4 = k^2 + 4k + 4 = (k+2)^2 = a_{k+2}$$

  which is what we wanted to show, and proves that $a_n = n^2$ for all even n.

- Case 2: k is odd:
  In this case we can let $m = \frac{k+3}{2}$ and $n = \frac{k+1}{2}$. Then $m + n = k + 2$, $m - n = 1$, $2m = k + 3$ and $2n = k + 1$. Notice that $k + 3$ and $k + 1$ are both even, so by case 1 above we know that $a_{k+3} = (k+3)^2$ and $a_{k+1} = (k+1)^2$. We also know $a_1 = 1$ be base case, thus:

$$a_{k+2} + 1 = \frac{(k+3)^2 + (k+1)^2}{2}$$

$$a_{k+2} = \frac{(k+3)^2 + (k+1)^2 - 2}{2} = \frac{2k^2 + 8k + 8}{2} = k^2 + 4k + 4 = (k+2)^2$$

  Which is what we wanted to show, and proves that $a_n = n^2$ for all odd n.

17 IC72

This solution was submitted only once.
Comments were received on the week of **October 31st**.
That same week it was given a C^3.

Strategies Used

- Find a penultimate step- try to find a sum for which a lot of terms cancel out.

Tactics Used

- Find and exploit symmetries- to find terms that cancel out.

Tools Used

- Factor tactic- to get an easy to telescope sum.

Question: Find a formula for

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3}+\cdots+\frac{1}{n(n+1)(n+2)}$$

For this answer we will use telescoping. Notice that the sum above can be expressed as

$$\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)}$$

furthermore, the fraction in the sum can be decomposed in such a way that :

$$\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)}=\sum_{k=1}^{n}\frac{1}{2}\Big(\frac{1}{k(k+1)}-\frac{1}{(k+1)(k+2)}\Big)$$

and because $\frac{1}{2}$ is just a constant we can pull it out and expand the sum to see that:

$$\frac{1}{2}\sum_{k=1}^{n}\Big(\frac{1}{k(k+1)}-\frac{1}{(k+1)(k+2)}\Big)=$$

$$\frac{1}{2}\Big(\Big(\frac{1}{1(2)}-\frac{1}{2(3)}\Big)+\Big(\frac{1}{2(3)}-\frac{1}{3(4)}\Big)+\Big(\frac{1}{3(4)}-\frac{1}{4(5)}\Big)+\cdots+\Big(\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\Big)\Big)$$

With this we can easily see that a a lot of terms will cancel by substraction:
(Note that this is not the actual sum, we omitted the $\frac{1}{2}$ to focus on the cancelation of terms, also known as telescoping):

$$\frac{1}{1(2)}+\Big(-\frac{1}{2(3)}+\frac{1}{2(3)}\Big)+\Big(-\frac{1}{3(4)}+\frac{1}{3(4)}\Big)+\Big(-\frac{1}{4(5)}+\frac{1}{4(5)}\Big)+\cdots+\Big(-\frac{1}{(n+1)(n+2)}\Big)=\frac{1}{1(2)}-\frac{1}{(n+1)(n+2)}$$

Therefore,

$$\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)}=\frac{1}{2}\sum_{k=1}^{n}\Big(\frac{1}{k(k+1)}-\frac{1}{(k+1)(k+2)}\Big)=\frac{1}{2}\Big(\frac{1}{1(2)}-\frac{1}{(n+1)(n+2)}\Big)=$$

$$\frac{1}{4}-\frac{1}{2(n+1)(n+2)}=\frac{2n^2+6n}{8n^2+24n+16}$$

which is the formula we were looking for.

18 IC92 (P)

This solution was submitted only once.
Comments were received on the week of **November 7**.
The comments said that id I fixed a minor detail, it would be C^3.

Strategies Used

- Get your hands dirty- find the first few examples to come up with a conjecture.

Tactics Used

- Invariance principle- the difference between any two summands is always either 0 or 1.

Tools Used

- Primes and divisibility- the division algorithm is a core element of the proof.

Question: Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$?

Following the guidelines above we can do some experimentation to find that there is

- one way to write 1: 1,
- two ways to write 2: 2, 1+1,
- three ways to write 3: 3, 2+1, 1+1+1,
- four ways to write 4: 4, 2+2, 2+1+1, 1+1+1+1,
- five ways to write 5: 5, 3+2, 2+2+1, 2+1+1+1, 1+1+1+1+1

and so on. We soon come to the conjecture that, following the guidelines above, there are n ways to write n.
To prove this, we can express n as the following sum:

$$n = a + a + \cdots + a + (a + 1) + (a + 1) + \cdots + (a + 1)$$

Then we can say that we have:

- k terms in total, where $1 \leq k \leq n$. Notice this range for k is true because we need to have at least one term, and all terms have to be integers so the maximum number of terms appears when we have n 1's.
- r terms of value $(a + 1)$, and $k - r$ terms of value a, where $0 \leq r \leq k$ because we need to have 0 or more a terms.

(continues on next page)

With this, we can express the sum above as $n = ak + r$. Now, let us restate that all numbers n, a, and k are defined to be positive integers and r is a non-negative integer. Then we can invoke the **division algorithm**, which states that for any pair of positive integers n and k where $n \geq k$, there exist *unique* non-negative integers a and r with $r \leq k$ such that $n = ak + r$ which is exactly what we have above.

Thus, if we take a fixed n we know that for each possible number of terms k there is just one choice of a and r that determine a unique way to express n in terms of a and $(a + 1)$. Moreover, there exist exactly n possible k's so there are exactly n unique ways to express n following the guidelines of the question.

19 IC99

This solution was submitted only once.
Comments were received on the week of **November 7**.
That same week it was given a C^3.

Strategies Used

- Formulate intermediate goals- find a desired coefficient in a generating function.

Tactics Used

- Find and exploit symmetries to know which exponents will get too big to be considered.

Tools Used

- Generating functions- to find the number of ways to choose the variables.
- Diophantine equations- find all integral solutions to a problem..

Question: How many solutions in natural numbers are there to the equation $a+b+c+d = 12$ **where** a **and** b **are odd?**

First of all, notice that because $a, b, c,$ and d must all be natural numbers, the maximum any of them can be is the number 9 (so that $9 + 1 + 1 + 1 = 12$). With this we have that a and b can be any number from the set $\{1, 3, 5, 7, 9\}$ (because they must be odd), while c and d can be any number from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
Moreover, we must choose the four numbers so that their sum is equal to 12. This is equivalent to saying that we must choose one term from the polynomial $(x + x^3 + x^5 + x^7 + x^9)$ (for a), one term from $(x + x^3 + x^5 + x^7 + x^9)$ (for b), one term from $(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9)$ (for c), and another one from $(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9)$ (for d), such that the sum of the exponents of all four terms is equal to 12.
Furthermore, we can find how many ways there are to get a sum of exponents equal to 12 by simply looking at the coefficient of x^{12} in the generating function

$$(x + x^3 + x^5 + x^7 + x^9)^2(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9)^2$$

Notice that when we expand each of the squares to get a multiplication of two polynomials, the term with the smallest exponent in both cases will be x^2. Because we are looking for exponents that add up to 12, any term with an exponent greater than 10 is of no interest to us. Therefore we have that the multiplication above is equal to

$$(x^2 + 2x^4 + 3x^6 + 4x^8 + 5x^{10} + \dots)(x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 7x^8 + 8x^9 + 9x^{10} + \dots)$$

(To see the complete work on how I got these expansions look at the attatched tables). Now, from above we can easily see that the terms that will multiply to get an exponent of x^{12} are:

$$x^2(9x^{10}) + 2x^4(7x^8) + 3x^6(5x^6) + 4x^8(3x^4) + 5x^{10}(x^2) =$$
$$9x^{12} + 14x^{12} + 15x^{12} + 12x^{12} + 5x^{12} =$$
$$55x^{12}$$

Hence, there are 55 possible natural number solutions to the equation $a + b + c + d = 12$.

20 Zeitz 1.3.14 (P)

I found this problem on the textbook and asked for permission to use it in one of my submissions. This solution was submitted once in November 7 but no comments were given back. It was submitted again by email around November 9, and an email response around November 11 confirmed it was C^3.

Strategies Used

- Wishful thinking- looking at what each individual term is less than to find a pattern.

Tactics Used

- None of those listed in the portfolio template.

Tools Used:

- Arithmetic and geometric sequences and series- this is an arithmetic series.
- Inequalities- question is about inequalities and they are used to find a contradiction.

Question: Let a_n be a sequence of positive real numbers such that $a_n \leq a_{2n} + a_{2n+1}$ for all n. Prove that $\sum_{n=1}^{\infty} a_n$ diverges.

Notice that by the properties described above,

- $a_1 \leq a_2 + a_3$,
- $a_2 \leq a_4 + a_5$,
- $a_3 \leq a_6 + a_7$,
- $a_4 \leq a_8 + a_9$, and so on.

Therefore we can conclude that overall:

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=2}^{\infty} a_n \implies a_1 + \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} a_n$$

Now, let us assume the contrary of what we are trying to prove, that is, let us assume that $\sum_{n=1}^{\infty} a_n$ converges. Then it must be that $\sum_{n=2}^{\infty} a_n$ also converges, so we can express it as $\sum_{n=2}^{\infty} a_n = S$, $S \in \mathbb{R}$. Then we can rewrite the above inequality as:

$$a_1 + S \leq S \implies a_1 \leq 0$$

However, our initial premise is that all terms a_n are *positive* real numbers, so we have reached a contradiction.
Hence, our assumption is wrong, and the sum $\sum_{n=1}^{\infty} a_n$ diverges.

21 IC121

This solution was submitted only once.
Comments were received on the week of **November 14**.
That same week it was given a C^3 with a suggestion, I don't remember if I did any changes.

Strategies Used

- Formulate intermediate goals- split the problem into two cases.

Tactics Used

- Find and exploit symmetries- each summand is symmetric with respect to $n/2$.

Tools Used

- Primes and divisibility- finding GCD of a pair of numbers to show they are relatively prime.
- Congruence- consider cases of congruence mod 4.

Question: Show that any natural number $n > 7$ can be expressed as a sum of two relatively prime numbers both greater than 1.
We will solve this problem considering the following two cases:

- Case 1: n is odd: In this case let us recall the rule that states that any two consecutive numbers are relatively prime. Because n is odd we know that $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ are consecutive numbers and hence, relatively prime. We also know that $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$ so we have found the two relatively prime numbers that we were looking for.
- Case 2: n is even: In this case we need to consider two more subcases, namely:
  - Case A: $n \equiv 0 \mod 4$: In this case $\frac{n}{2}$ will be even so $\frac{n}{2} - 1$ and $\frac{n}{2} + 1$ will be odd. Our claim is that $\frac{n}{2} - 1 + \frac{n}{2} + 1 = n$ is the sum that we are looking for. Let $k = \frac{n}{2} - 1$ and $k + 2 = \frac{n}{2} + 1$. We need to show that for any odd k, k and $k + 2$ are relatively prime. That is, $GCD(k, k+2) = 1$ so we have to show that they have no common prime divisors.
    Now because the two numbers are odd, we know neither of them is divisible by 2. Furthermore, for any other prime number $p > 2$ if $p \mid k$ then the next smallest number after k that p divides is $k + p$ but $p > 2$ so $k + p > k + 2 \implies p \nmid k + 2$. Similarly if $p \mid k + 2$ the previous biggest number that p divides is $k + 2 - p < k \implies p \nmid k$.
    So $\frac{n}{2} - 1$ and $\frac{n}{2} + 1$ are relatively prime.
  - Case B: $n \equiv 2 \mod 4$: In this case $\frac{n}{2}$ will be odd so $\frac{n}{2} - 2$ and $\frac{n}{2} + 2$ will be odd. Then, in a similar way as above we must show that for any odd k, $GCD(k, k+4) = 1$, i.e. that they have no common prime divisors. Once again because they are both odd, 2 is not a common divisor.
    Next if we take $p = 3$, on the one hand if $p \mid k$ then $p \mid k + 3$ and $p \mid k + 6$ but $p \nmid k + 4$. On the other hand if $p \mid k + 4$ then $p \mid k + 1$ and $p \mid k - 2$ but $p \nmid k$. Furthermore, any other prime $p > 3$ will follow a similar pattern as in case A so $GCD(k, k + 4) = 1$ and $\frac{n}{2} - 2 + \frac{n}{2} + 2$ are relatively prime.

Therefore, in all cases a natural number $n > 7$ can be expressed as a sum of two relatively prime numbers.

22 IC118

This solution was submitted only once.
Comments were received on the week of **November 14**.
That same week it was given a C^3.

Strategies Used

- Get your hands dirty- before coming up with the recursive relations I had to draw many $3 \times n$ rectangles to find a pattern.

Tactics Used

- Invariance principle- There are only four states each of which yield to the one of the same four states.

Tools Used

- Recurrence relations- two of them.

Question: How many ways are there to tile a $3 \times n$ rectangle with 2×1 tiles?
To solve this problem we will come up with a recursive function that finds a_n = the number of tilings of a $3 \times n$ rectangle using 2×1 tiles. To do this, we will use a helper recursive function that finds b_n = the number of tilings of a $3 \times n$ rectangle *with one square removed* using 2×1 tiles. To come up with these functions, we will first consider the following four base cases:

- The rectangle for $n = 1$ has 0 possible tilings. That is $a_1 = 0$.
- The rectangle for $n = 2$ has 3 possible tilings. That is $a_2 = 3$.
- The rectangle for $n = 1$ with one square removed has 1 possible tilling. That is $b_1 = 1$.
- The rectangle for $n = 2$ with one square removed has 0 possible tilings. That is $b_2 = 0$.

Now, keeping these base cases in mind, we can take any $3 \times n$ rectangle with $n > 2$. Then to find a_n we have to count all the possibilities in the following two cases:

- Case 1: There are three vertical tiles touching the nth row. In this case, we can "remove" the two last rows, and count all the possible tilings for a $3 \times n - 2$ rectangle, that is a_{n-2}.
- Case 2: There is one vertical tile and one horizontal tile touching the nth row. In this case, we can also "remove" the row and the square covered by the tiles. Notice that there are two ways to do this, one when the vertical tile is on the first column, and one when the vertical tile is on the third column. Therefore we have to count twice the number of possible tilings of a $3 \times n - 1$ rectangle with one square removed, that is $2b_{n-1}$.

From the cases above, we can come up with the recursive formula $a_n = a_{n-2} + 2b_{n-1}$. However, it still remains to define a recursive function for b_n. In order to do this, consider the following two more cases of a $3 \times n$ rectangle with one square removed:

- Case 1: There is a horizontal tile covering the two "orphan" squares (that is, the two squares in the row from which the one square was removed). In this case we can "remove" those two squares, and count the number of tilings of a $3 \times n - 1$ rectangle. That is a_{n-1}.
- Case 2: There are two vertical tiles covering the two "orphan" squares. In this case we can "remove" those four squares, which will yield a single "orphan" square, this square can only have a vertical tile, which when removed will result in a new $3 \times n - 2$ rectangle with one square removed. That is b_{n-2}.

Hence we have that $b_n = a_{n-1} + b_{n-2}$ which is what we were missing.

23 OC78

This solution was submitted only once.
Comments were received on the week of **November 14**.
It did NOT receive a C^3, however, I have done the corrections suggested in the comments.

Strategies Used

- Formulate intermediate goals- find GCD of certain Fibonacci numbers.

Tactics Used

- Extreme principle- find the largest p can be.

Tools Used

- Recurrence relations- Fibonacci.
- Primes and divisibility- To show two numbers are relatively prime.

Question: Prove that consecutive Fibonacci numbers are always relatively prime.

Let the nth Fibonacci number be denoted by f_n, then to show that consecutive Fibonacci numbers are always relatively prime we must show that $GCD(f_n, f_{n+1}) = 1$ is true for all $n > 1$. To do this we will use proof by induction on n.
First, our base case will be when $n = 1$ then $f_n = 1$ and $f_{n+1} = 1$, and we know $GCD(1, 1) = 1$.
Next, for our induction hypothesis we will assume that $GCD(f_k, f_{k+1}) = 1$ is true for some $n = k$.
Then, for the inductive step we need to show that for $n = k+1$, $GCD(f_{k+1}, f_{k+2}) = 1$ is also true.
Notice this is equivalent to showing that $GCD(f_{k+1}, f_k + f_{k+1}) = 1$. That is, we want to show for any integer p that if both $p \mid f_{k+1}$ and $p \mid f_k + f_{k+1}$ then the largest possible value of p is 1.
Now, if $p \mid f_{k+1}$ and $p \mid f_k + f_{k+1}$, it follows that $p \mid f_k$. Moreover, from out induction hypothesis we have that $GCD(f_k, f_{k+1}) = 1$, therefore, the largest p can ever be is 1, which is what we wanted to show.
Hence, all consecutive Fibonacci numbers are relatively prime.

24 IC122

This solution was submitted only once.
Comments were received on the week of **November 14**.
That same week it was given a C^3.

Strategies Used

- Formulate a penultimate step- suffices to show not congruent to 0 or 1 mod 4.

Tactics Used

- Invariance principle- a perfect square is always congruent 0 or 1 mod 4.

Tools Used

- Congruence- all the proof is based on congruence.

Question: Let a be a natural number. Prove that $a^2+(a+1)^2+(a+2)^2+(a+3)^2+(a+4)^2$ is never a perfect square.
To do this first notice that in general, for any integer k one of the following four cases must be true:

- $k \equiv 0 \mod 4 \implies k^2 \equiv 0 \mod 4$
- $k \equiv 1 \mod 4 \implies k^2 \equiv 1 \mod 4$
- $k \equiv 2 \mod 4 \implies k^2 \equiv 4 \mod 4 \implies k^2 \equiv 0 \mod 4$
- $k \equiv 3 \mod 4 \implies k^2 \equiv 9 \mod 4 \implies k^2 \equiv 1 \mod 4$

Hence, for any integer k, the perfect square k^2 must be congruent to either $0 \mod 4$ or $1 \mod 4$. So if we want to show that some number m is NOT a perfect square, it suffices to show that m is NOT congruent $0 \mod 4$ nor $1 \mod 4$. Furthermore, notice that consecutive perfect squares alternate between being congruent $0 \mod 4$ and being congruent $1 \mod 4$.
Now, let $a^2+(a+1)^2+(a+2)^2+(a+3)^2+(a+4)^2 = m$. We want to show that m is never a perfect square, therefore we must show that in all cases either $m \equiv 2 \mod 4$ or $m \equiv 3 \mod 4$. Next, because all the terms in the sum above are perfect squares, we can consider the following two cases:

- $a^2 \equiv 0 \mod 4$. This implies all of the following:
  - $(a+1)^2 \equiv 1 \mod 4$,
  - $(a+2)^2 \equiv 0 \mod 4$,
  - $(a+3)^2 \equiv 1 \mod 4$, and
  - $(a+4)^2 \equiv 0 \mod 4$. Hence
  - $m \equiv (0+1+0+1+0) \mod 4 = 2 \mod 4$.
- $a^2 \equiv 1 \mod 4$. Following a similar pattern as above, this implies that $m \equiv (1+0+1+0+1) \mod 4 = 3 \mod 4$

So overall, in all cases m is either congruent with $2 \mod 4$ or with $3 \mod 4$, therefore m can never be a perfect square, which is what we wanted to show.

25 IC131 (P)

This solution was submitted only once. I am aware it was presented in class but I turned it in that same day before seeing the presentation.
Comments were received on the week of **November 21st**.
That same week it was given a C^3, along with a suggestion, however I did not make any further changes.

Strategies Used

- Get your hands dirty- tried a lot of values of n until we found a pattern for the necessary condition.

Tactics Used

- Invariance principle- the sum of all integers up to n must always be divisible by 3.

Tools Used

- Partitions and bijection- we are looking for partitions in the problem.
- Primes and divisibility- the divisibility of the sum of the elements in the set is important to solve the problem.
- Congruence- congruence $\mod 6$ is used.

Question: For what values of n can $\{1, 2, \ldots, n\}$ be partitioned into three subsets with equal sums?

First of all notice that if we partition a set $A = \{1, 2, \ldots, n\}$ into 3 subsets of equal sums, each of these sums must be exactly

$$\frac{\sum_{a \in A} a}{3}$$

Then, because all elements in A are integers, it follows that the only way in which this can be possible is if

$$3 \;\Big|\; \sum_{a \in A} a$$

Furthermore, $n = 1, 2$ and 3 are too small to be able to satisfy the initial condition, and $1+2+3+4 = 10$ is not divisible by 3, therefore it must also be true that $n \geq 5$. Below we can see the first four instances of this pattern:

(continues next page)

- $n = 5$, $1 + 2 + 3 + 4 + 5 = 15$, $15/3 = 5$, the partition to three subsets with sum equal to 5 is $\{5\}, \{4, 1\}, \{3, 2\}$
- $n = 6$, $15 + 6 = 21$, $21/3 = 7$, the partition to three subsets with sum equal to 7 is $\{6, 1\}, \{5, 2\}, \{4, 3\}$
- $n = 8$, $21 + 7 + 8 = 36$, $36/3 = 12$, the partition to three subsets with sum equal to 12 is $\{8, 4\}, \{7, 5\}, \{6, 3, 2, 1\}$
- $n = 9$, $36 + 9 = 45$, $45/3 = 15$, the partition to three subsets with sum equal to 15 is $\{9, 5, 1\}, \{8, 7\}, \{6, 4, 3, 2\}$

Taking the four values of n above, notice that $5 \equiv 5 \mod 6$, $6 \equiv 0 \mod 6$, $8 \equiv 2 \mod 6$ and $9 \equiv 3 \mod 6$. Our claim is that in all cases where the desired partition is possible, n must be congruent to $0, 2, 3$ or $5 \mod 6$. We will prove this by induction:
Take the four examples above as base cases. Then assume that for some k, the set $\{1, 2, \ldots, k\}$ can be partitioned into three subsets with equal sums. Now we can show that this must also be the case for $k + 6$. Consider the set $\{1, 2, \ldots, k + 6\}$, and partition it in two sets $\{1, 2, \ldots, k\}$ and $\{k + 1, k + 2, k + 3, k + 4, k + 5, k + 6\}$. By our induction hypothesis we know that the first set can be partitioned into three sets such that the sum of the elements of each of them is equal to some number b. Moreover, we can easily partition the second set into the three subsets $\{k + 1, k + 6\}, \{k + 2, k + 5\}, \{k + 3, k + 4\}$, such that the sum of each of the subsets is equal to $2k + 7$. Hence, the whole set $\{1, 2, \ldots, k + 6\}$ can be partitioned into three sets with equal sum, namely $2k + 7 + b$.

Now, the induction above has shown how for all $n \equiv 0, 2, 3, 5 \mod 6$ it is possible to obtain the desired partition. However, it remains to show that when $n \equiv 1, 4 \mod 6$ it is NOT possible to get such a partition. To do so we will use contradiction: First consider some $n \equiv 1 \mod 6$, and notice that this means that $n^2 \equiv 1 \mod 6$ and $n + n^2 \equiv 2 \mod 6$. In a similar way, consider some $n \equiv 4 \mod 6 \implies n^2 + n \equiv 20 \mod 6 = 2 \mod 6$ also.
Next assume that the set $\{1, 2, \ldots, n\}$ can be partitioned into three sets of equal sum $b \in \mathbb{Z}$, and that

$$3b = \sum_{a=1}^{n} a = \frac{n(n+1)}{2} \implies$$

$$\frac{n^2 + n}{2} = 3b \implies n^2 + n = 6b \implies n^2 + n \equiv 0 \mod 6$$

However, we had already established that $n^2 + n \equiv 2 \mod 6$ so we have a contradiction. So it is impossible to get a partition of three sets of equal sum any time that $n \equiv 1, 4 \mod 6$ which is what we wanted to show.

26 IC139 (P)

This solution was submitted only once.
Comments were received on the week of **November 21st**.
That same week it was given a C^3.

Strategies Used

- Formulate intermediate goals- look at quadrilateral $OABC$ and attempt to find r there.

Tactics Used

- Find and exploit symmetries- of the isoceles triangles as well as of the quadrilateral.

Tools Used

- Geometry- Triangles, angles, and cosine rule.

Question: A hexagon inscribed in a circle has three consecutive sides of length a and three consecutive sides of length b. Determine the radius of the circle in terms of a and b

(For this paragraph see fig. 1) Call the center of the circle O and its radius r, then draw six radii connecting O to each of the vertices of the hexagon. Notice that this yields to six isoceles triangles with two sides length r, and the other side either length a or length b. Furthermore, by SSS all the triangles with a side a are congruent, so each of their angles at O measure $\alpha°$, and each of their angles not at O measure $((180 - \alpha)/2)°$. Similarly all triangles with a side b are also congruent, so each of their angles at O measure $\beta°$, and each of their angles not at O measure $((180 - \beta)/2)°$.
(For this paragraph see fig. 2) Next, notice that all the angles at O must add up to 360°, so we have the equation $3\alpha + 3\beta = 360 \implies \alpha + \beta = 120$. Now label three consecutive vertices of the hexagon A B and C such that $|AB| = a$ and $|BC| = b$, and take the quadrilateral $OABC$. Then notice that angle $AOC = \alpha + \beta = 120$. Furthermore we can find that angle $ABC =$

$$\theta = \frac{180-\alpha}{2} + \frac{180-\beta}{2} = \frac{180-\alpha+180-\beta}{2} = \frac{360-\alpha-\beta}{2} \implies$$
$$-\theta = \frac{\alpha+\beta-360}{2} = \frac{120-360}{2} = \frac{-240}{2} = -120 \implies$$
$$\theta = 120$$

So $AOC = ABC = 120$, $|AB| = a$, $|BC| = b$, $|OA| = |OB| = r$. Moreover, it is useful to know that $\cos(120) = -1/2$. Finally, draw a line AC and call $|AC| = c$, then we can use the cosine rule to write the following two equations:

$$c^2 = a^2 + b^2 - 2ab\cos(120) \qquad \textit{and} \qquad c^2 = r^2 + r^2 - 2r^2\cos(120)$$
$$\implies a^2 + b^2 - 2ab\cos(120) = 2r^2 - 2r^2\cos(120)$$
$$\implies a^2 + b^2 + ab = 2r^2(1 + \frac{1}{2}) \implies a^2 + b^2 + ab = 3r^2$$
$$\implies r = \sqrt{\frac{a^2+b^2+ab}{3}}$$

Which is the equation we were looking for.

27 IC143

This solution was submitted only once.
Comments were received on the week of **November 21st**.
It did NOT obtain a C^3 but I have made the corrections in the comments.

Strategies Used

- Wishful thinking- imagine a triangle with the desired characteristics and try to see if it is possible that it exists.

Tactics Used

- None of those listed in the portfolio template.

Tools Used

- Inequalities- triangle inequality.
- Geometry- area of a triangle.

Question: Is it possible for a triangle to have altitudes equal to 6, 10 and 20.

The answer to this question is no. To prove this, assume there is a triangle with sides a, b, and c and altitudes 6, 10 and 20 respectively. Then the area of the triangle is

$$A = \frac{6a}{2} = \frac{10b}{2} = \frac{20c}{2} \implies$$

$$3a = 10b = 20c$$

Now, if we express b and c in terms of a we have that
$3a = 10b \implies \frac{3a}{10} = b$ and
$3a = 20c \implies \frac{3a}{20} = c$.
Furthermore we have that $b + c = \frac{3a}{10} + \frac{3a}{20} = \frac{6a+3a}{20} = \frac{9a}{20}$.
Next, notice that $20a \geq 9a$ because a is positive. Then $a \geq \frac{9a}{20} \implies a \geq b + c$. However, the triangle inequality states that $a < b + c$, therefore we have reached a contradiction.
Hence, there cannot be a triangle with altitudes 6, 10 and 20 which is what we wanted to show.

28 OC88

This solution was submitted only once.
Comments were received on the week of **November 21st**.
That same week it was given a C^3.

Strategies Used

- Specialization- start by looking at the example where $b = 2$.
- Get your hands dirty- plug all possible values of b in a calculator to see which ones are valid.

Tactics Used

- Extremal principle- look at the smallest possible solution.

Tools Used

- Polynomials- solving quadratic equations.
- Diophantine equations- looking for integer solutions.

Question: Find all pairs of nonnegative integers (x, y) **such that** $x^3 + 8x^2 - 6x + 8 = y^3$.

First of all, notice that $8x^2 - 6x + 8$ is positive for all non-negative x because $8x^2$ grows faster than $6x$. This means that $x^3 \leq y^3 \implies x \leq y$. Therefore, we can say that $y = x + b$ where $b \in \mathbb{N}$ is the difference between x and y, and $y^3 = x^3 + 3x^2b + 3xb^2 + b^3$.
Now, looking at the initial equation we can set $y^3 =$

$$x^3 + 8x^2 - 6x + 8 = x^3 + 3x^2b + 3xb^2 + b^3 \implies$$

$$8x^2 - 6x + 8 - 3x^2b - 3xb^2 - b^3 = 0$$

$$= (8 - 3b)x^2 - (6 + 3b^2)x + 8 - b^3 = 0$$

Next, for each value of b that yields to a real, integer solution of the quadratic equation above, we will have that $(x, y) = (x, x + b)$ is one of the pairs we are looking for. For example, when $b = 2$ we have

$$(8 - 3b)x^2 - (6 + 3b^2)x + 8 - b^3 = 0$$

$$= (8 - 6)x^2 - (6 + 12)x + 8 - 8 = 0$$

$$= 2x^2 - 18x = 0$$

And we can use the quadratic formula to get:

$$x = \frac{18 \pm \sqrt{18^2}}{2(2)}$$

So the solutions are

- $x = 0$ and then $x + b = 0 + 2 = 2$
- $x = 9$ and then $x + b = 9 + 2 = 11$

which means the pairs $(0, 2)$ and $(9, 11)$ satisfy $x^3 + 8x^2 - 6x + 8 = y^3$:

$$0^3 + 8(0)^2 - 6(0) + 8 = y^3 \implies y = 2$$

and

$$9^3 + 8(9)^2 - 6(9) + 8 = y^3 \implies 729 + 648 - 54 + 8 = y^3 \implies$$
$$1331 = y^3 \implies y = 11$$

Now this is just the instance where $b = 2$, it remains to determine for which values of b, the quadratic equation above has real and integer solutions. Notice that for the solution of a quadratic equation to be real, its discriminant must be positive. The discriminant of our quadratic equation is $(6+3b^2)^2 - 4(8-3b)(8-b^3)$. If we plug this in Wolfram Alpha we can find that the integer values of b for which this determinant is positive lie between 2 and 11 (inclusive), furthermore, again with the help of a calculator we can see that for any $b > 2$ the determinant is not an integer. Thus, the only real, integer solutions of our quadratic formula are the ones that we already found, 0 and 9. Hence, the only pairs (x, y) for which $x^3 + 8x^2 - 6x + 8 = y^3$ are (0.2) and $(9, 11)$.

29 IC145

This solution was submitted only once.
Comments were received on the week of **November 30**.
That same week it was given a C^3.

Strategies Used

- Formulate intermediate goals- get the area of the smaller triangles.

Tactics Used

- Find and exploit symmetries- sum of areas and total area have many similar terms that cancel out.

Tools Used

- Geometry- area of a triangle.

Let P be an arbitrary point in the interior of an equilateral triangle. Prove that the sum of the distances of P to the three sides is equal to the altitude of the triangle.

Let the vertices of the triangle be A, B, and C, also let D, E and F be the intersections of the lines passing through P and perpendicular to AB, BC, and CA respectively, finally let $s = |AB| = |BC| = |CA|$.
Now, we are trying to show that $|PD|+|PE|+|PF| = h$ where h is the altitude of the triangle. To prove this we draw three more lines PA, PB, and PC. Then consider the triangle ABP and notice that it has altitude $|PD|$ and base $|AB| = s$ so its area is $s|PD|/2$. Similarly, the area of triangles BCP and CAP are $s|PE|/2$ and $s|PF|/2$ respectively. Notice that the sum of these areas must be the area of the whole triangle, thus

$$s|PD|/2 + s|PE|/2 + s|PF|/2 = sh/2 =$$

$$\frac{s}{2}(|PD| + |PE| + |PF|) = \frac{s}{2}(h) \implies$$

$$|PD| + |PE| + |PF| = h$$

which is what we wanted to show.

30 IC151

This solution was submitted only once.
Comments were received on the week of **November 30**.
That same week it was given a C^3.

Strategies Used

- Find a penultimate step- the value of h found in the previous proof is the penultimate step.

Tactics Used

- Invariance principle- h is always the same with respect to s.

Tools Used

- Geometry- height of a triangle.

Let triangle ABC be an equilateral triangle and P an arbitrary point within the triangle. Perpendiculars PD, PE and PF are drawn to the three sides of the triangle. Show that no matter where P is chosen

$$\frac{|PD| + |PE| + |PF|}{|AB| + |BC| + |CA|} = \frac{1}{2\sqrt{3}}$$

Let $s = |AB| = |BC| = |CA|$ and h be the altitude of the triangle. Then by the proof immediately above (IC145) what we are trying to show is equivalent to showing that

$$\frac{h}{3s} = \frac{1}{2\sqrt{3}}$$

Now, by definition the altitude of an equilateral triangle is

$$h = \frac{s\sqrt{3}}{2} \implies \frac{h}{3s} = \frac{s\sqrt{3}}{2} \div 3s = \frac{s\sqrt{3}}{6s} = \frac{\sqrt{3}}{6}\frac{\sqrt{3}}{\sqrt{3}} = \frac{3}{6\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

Which is what we wanted to show.